Lecture 6

Tuesday, 27 February 2024 9:03 AM

Integrality: TUM & TDI

When can we obtain an integral soln to an LP?

- A matrix / vector is integral if all entries are integers.
- $P = \{x \in \mathbb{R}^n : Ax \le b\}$ is integral if all its vertices are integral.

Theorem: If P is bounded & integral, & $c \in \mathbb{R}^n$ is rational, then we can in polynomial time obtain an integral soln to $\max\{c^T x : x \in P\}$.

Proof: Use ellipsoid to obtain a (possibly fractional) soln to $\max\{c^T x : x \in P\}$, then add constraints until you reach a vtx.

So when is a polyhedron P integral?

Defn: ① $A \in \mathbb{Z}^{m \times m}$ is unimodular if $\det(A) \in \{-1, +1\}$

② $A \in \mathbb{Z}^{m \times n}$ is totally unimodular if every square submatrix B of A has $\det(B) \in \{-1, 0, 1\}$.

(Also if A is TUM then $A \in \{-1, 0, +1\}^{m \times n}$)

E.g. $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is TUM

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ is not TUM, $\det(A) = 2$.

Theorem: Let A be TUM & $b \in \mathbb{Z}^m$. Then $P = \{x : Ax \le b\}$ is integral.

Proof: Let x^* be a vtx. Then there are n L.I. constraints tight at x^*, say $a_1^T x = b_1, \dots, a_n^T x = b_n$.

Let $B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $b' = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

Then $x^* = B^{-1} b'$.

By Cramer's rule, $x^*_j = \frac{\det B^j}{\det B}$

where $B^j = [B_1, \dots, b', \dots, B_n]$

Now $\det B^j \in \mathbb{Z}$, $\det(B) \in \{-1, +1\}$ (since nonsingular)

Hence x^* is integral. ∎

(or directly $B^{-1} = \frac{C^T}{\det(B)}$, all cofactors are integral, $b' \in \mathbb{Z}^n$, $\det(B) \in \{-1, +1\}$)

Thus if TUM, $b \in \mathbb{Z}^m$, $c \in \mathbb{R}^n$ is rational, can find an integral optimal soln efficiently.

Claim: If $A \in \mathbb{R}^{m \times n}$ TUM, then so are:

① $-A$

② A^T

③ $[A \; I]$

④ $[A \; 0]$

⑤ $\begin{bmatrix} A \\ -A \end{bmatrix}$

⑥ $[A \; -A]$

Can show that $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ is TUM

(or in general, multiplying a row/col by -1)

(prove yourself)

Corollary: If A is TUM & $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, then both polyhedra $P = \{x : Ax \le b\}$ & $D = \{y : A^T y = c, y \ge 0\}$ are integral.

Corollary: If A is TUM & a, b, c, d are integral vectors, the polyhedron $P = \{x : c \le x \le d, a \le Ax \le b\}$ is integral.

So how do you show a matrix is TUM?

Theorem: Let $A \in \mathbb{R}^{m \times n}$. Then A is TUM iff any subset S of rows can be partitioned into sets S_1, S_2 s.t.

$$\sum_{i \in S_1} a_i - \sum_{i \in S_2} a_i \in \{-1, 0, +1\}^n$$

(proof skipped)

APPLICATIONS:

1. Given a digraph G, consider the $|V| \times |E|$ node-arc incidence matrix:

$$M_{v,e} = +1 \text{ if } e \in \delta^+(v)$$
$$= -1 \text{ if } e \in \delta^-(v)$$
$$= 0 \text{ o.w.}$$

E.g.

$M = $

	e_1	e_2	e_3
u	+1	0	+1
v	−1	+1	0
w	0	−1	−1

Claim: M is TUM.

Proof: Will use Ghouila-Houri characterization to show M is TUM.

For any subset S of rows (i.e. $S \subseteq V$), need to show $\exists S_1, S_2$ s.t. $\sum_{i \in S_1} m_i - \sum_{i \in S_2} m_i \in \{-1, 0, +1\}^{|E|}$

Take $S_1 = S$, $S_2 = \phi$. Then $\forall e$, $\sum_{i \in S_1} m_{i,e} \in \{-1, 0, 1\}$ as reqd.

Now consider the LP for max flow:

$$\max \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e$$

s.t. $\forall v \ne s, t$, $\boxed{\sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} x_e = 0}$ ← this is the node-arc incidence matrix, w/ rows for s & t removed

$\forall e$ $\quad c_e \ge x_e \ge 0$ → identity constraints

hence this polytope is integral (A is TUM, $b \in \mathbb{Z}^m$) (assuming all capacities are integral.

hence there exists an integral max-flow.

(& similarly the dual has an integral soln.)

What about node-arc incidence matrices for undirected graphs?

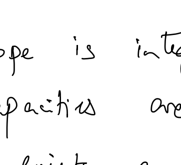
Eg.

$M = $

u	1	1	0
v	1	0	1
w	0	1	1

is not TUM

(compare to



)

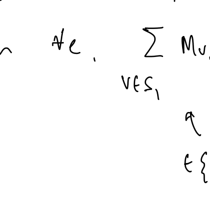
2. BIPARTITE GRAPHS.

Let $G = (A \cup B, E)$ be an undirected bipartite graph

Consider the incidence matrix M.

$$M_{v,e} = 1 \text{ if } e \text{ is incident on } v$$
$$= 0 \text{ o.w.}$$

Eg.



$M = $

a	1	1	0
b	0	0	1
c	1	0	1
d	0	1	0

Claim: M is TUM.

Proof: Given $S \subseteq V$, let $S_1 = S \cap A$, $S_2 = S \cap B$.

Then $\forall e$, $\underbrace{\sum_{v \in S_1} M_{v,e}}_{\in \{0,1\}} - \underbrace{\sum_{v \in S_2} M_{v,e}}_{\in \{0,1\}} \in \{-1, 0, 1\}$ as required. ∎

Consider the problem of finding a maximum cardinality matching in a b.p. graph:

$$\max \sum_e x_e \qquad \Big| \qquad \min \sum_v y_v$$

s.t. $\boxed{\forall v, \; \sum_{e \in \delta(v)} x_e \le 1}$ $\qquad$ $\forall e$ $\quad y_u + y_v \ge 1$

$\forall e$ $\quad x_e \ge 0$ $\qquad$ $\forall v$ $\quad y_v \ge 0$

node-arc incidence matrix

Hence in this case, can directly get a max cardinality matching.

The dual for the LP corresponds to (fractional) vertex cover.

By duality (& since there is an integral optimal soln), for bipartite graphs, max matching = min vertex cover.

This is known as Konig's theorem.

(not true for non-b.p. graphs).

Lastly:

Claim: If $A \in \{0,1\}^{m \times n}$ has the consecutive 1's property, i.e., in any row, all the 1's are consecutive, then A is TUM.

(Prove yourself)

A weaker condn. for a polyhedron to be integral is given by Total Dual Integrality.

This is a bit tricky, & not commonly used:

Defn: A system of inequalities $Ax \le b$ is TDI if $\forall c \in \mathbb{Z}^n$, $\exists$ an integral dual optimal soln. y^*, whenever $\exists$ an optimal soln to the dual.

i.e. $\quad \min b^T y$ $\quad$ has an integral optimal soln,

$\quad A^T y = c$ $\quad$ if there is an optimal soln.

$\quad y \ge 0$

note that if A is TUM, $Ax \le b$ is TDI for all $b \in \mathbb{R}^m$.

Theorem: If $Ax \le b$ is TDI <u>and</u> b is integral, then $P = \{x : Ax \le b\}$ is integral

NOTE: being TDI is a property of the system of inequalities, NOT the polyhedron.

Eg. Consider:

$x + 2y \le 6$

$2x + y \le 6$

$x, y \ge 0$

$c = (1,1)$

dual: $\min 6y_1 + 6y_2$

$2y_1 + y_2 = 1$

$y_1 + 2y_2 = 1$

$y_1, y_2 \ge 0$

only feasible soln. is $y_1 = y_2 = 1/3$.

but if we add the constraint

$x + 2y \le 6$

$2x + y \le 6$

$x + y \le 4$

$x, y \ge 0$

$c = (1,1)$

dual: $\min 6y_1 + 6y_2 + 4y_3$

$y_1 + 2y_2 + y_3 = 1$

$2y_1 + y_2 + y_3 = 1$

$y_1, y_2, y_3 \ge 0$

$(0,0,1)$ is an integral optimal soln.

Theorem: Any polyhedron P has a representation $P = \{x : Ax \le b\}$ such that:

① $Ax \le b$ is TDI

② A is integral

However b is integral iff P is integral.

Can be used to give a poly-time algo for max matchings in general graphs:

$$\max \sum_e x_e$$

$\forall v$ $\quad \sum_{e \in \delta(v)} x_e \le 1$

$\forall S \subseteq V : |S|$ is odd

$\quad \sum_{e \in E(S)} x_e \le \left\lfloor \frac{|S|}{2} \right\rfloor$ → odd-set constraints

$\forall e$ $\quad x_e \ge 0$

Theorem: The system of inequalities is TDI, & is equivalent to the convex hull of all (integral) matchings.

(proof skipped)